Goethe-Center for Scientific Computing (G-CSC)
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Modelling and Simulation I

(Practical SIM1, WS 2016/17)

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Exercise sheet 3 (Due: Mo., 21.11.2016, 10h)

Exercise 1 (5P)
Implement the explicit euler method in order to solve the system of ordinary differential equation (ODE)

$$\begin{cases} \text{Find } \mathbf{u} : [t_0, t_n] \mapsto \mathbb{R}^d, \text{ such that} \\ \frac{\partial}{\partial t}\mathbf{u}(t) = \mathbf{f}(t, \mathbf{u}) \qquad \text{on } [t_0, t_n], \\ \mathbf{u}(t_0) = \mathbf{u}_0, \end{cases} \tag{1}$$

where $\mathbf{u}_0 \in \mathbb{R}^d$ is the start value and $t_0, t_n \in \mathbb{R}$ are the start- and endpoints of the interesting time interval.

Hints:

- Download the `vectoroderhsinterface.jar` plugin, install it via

  `Plugins -> Install Plugin`

  restart VRL-Studio and add the plugin to the project via

  `File -> Select Plugins`

- In order to plot the solution, use the `VectorTrajectoryPlotter` and the `SimplePlotter` that are provided by the plugin.
- Use the `VectorTrajectory` to store the computed values $(t_i, \mathbf{u}_i)$ inside `VectorExplicitEuler` and return it as `ArrayList<double[]>`
- use `@ParamInfo(style = "array") Double[] u0` to allow specification of start values as arrays.
- Pass the user-defined right-hand side using the `VectorRhsODEInterface` interface.
- Having created a `VectorRhsODE` instance, this can be passed to the euler component using a data connection starting at the upper right corner of the rhs object.

- Specify the rhs using groovy code, e.g. `f = [u[1], u[0]]`. Remember the groovy notation: $x^p \to$ `x**p`, $\sqrt{x} \to$ `sqrt(x)`.

Exercise 2 (2 P + 1P + 3P + 3P)
Lets consider the populations of two animal species: a population of prey N (e.g. rabbits) and a population of predator P (e.g. foxes) and study the evolution of both species in time. We assume that the population of prey is supplied with infinite food recources (e.g. grass) and reproduces itself by a growth rate $\gamma > 0$. The population of predator in absence of prey will decrease with a mortality rate $\delta > 0$. The food for the predator is the population of prey, i.e. we assume that whenever two individuals of the two species meet the prey will be eaten by the predator with a certain probability described by the searching efficiency $s > 0$. Finally, the predator's efficiency at turning food into offspring (conversion efficiency) is given by $c > 0$. Thus, the net effect of reproduction of the predator is described by the reproduction rate $r := s \cdot c > 0$.
The mathematical model for this system is given by the famous Lotka-Volterra equations, namely,

$$\begin{cases} \text{Find } N, P : [t_0, t_n] \mapsto \mathbb{R}, \text{ such that} \\ N'(t) = \gamma \cdot N - s \cdot N \cdot P, \\ P'(t) = -\delta \cdot P + r \cdot N \cdot P, \\ N(0) = N_0, \\ P(0) = P_0. \end{cases} \tag{2}$$

Tasks/Questions:

(a) Use the explicit Euler scheme with $h = 0.001$ in order to solve the Lotka-Volterra model with $\gamma = 1.5, \delta = 2, s = 0.01, c = 0.8$ and $N_0 = 200, P_0 = 100$ for $t \in [0, 30]$. Produce a plot. What behaviour between prey and predator do you observe?

(b) A stationary state is present when both populations do not change in time. This can be expressed by the condition $\partial_t N = 0, \partial_t P = 0$. What are the two solutions for the stationary state? Provide a plot for the non-trivial solution.

(c) Lets refine the model (2): The population growths of the prey population will be inhibited when overpopulated. Take this into account using an inhibitation factor 0.001. Provide a plot for $t \in [0, 30]$. What is the qualitative behaviour? What is the non-trivial stationary state for the prey population (without proof)?

(d) Back to the pure model (2) (i.e. neglecting the inhibitation factor): Lets assume a third species Q (e.g. a wolf) is involved in the system. This species is a predator of the species P but does not interact with the population N. Model this system assuming a search efficiency $s_Q = 0.02$, a conversion efficiency $c_Q = 0.4$ and an initial population $Q_0 = 100$. Provide a plot for a death rate $\delta_Q = 1, 1.2, 1.5$. What is the qualitative behaviour of the population Q in the three cases?

Exercise 3 (1 P + 4P + 1P)
The motion of bodies in space (e.g. planets, space crafts) is due to the gravitational force. For the two-body problem, this is described using Newton's second law, i.e.,

$$m_1 \frac{\partial^2 \mathbf{x}_1}{\partial t^2} = -G m_2 m_1 \frac{\mathbf{x}_1 - \mathbf{x_2}}{||\mathbf{x}_1 - \mathbf{x_2}||_2^3}, \tag{3}$$

$$m_2 \frac{\partial^2 \mathbf{x}_2}{\partial t^2} = -G m_1 m_2 \frac{\mathbf{x}_2 - \mathbf{x_1}}{||\mathbf{x}_2 - \mathbf{x_1}||_2^3}, \tag{4}$$

$$\tag{5}$$

where $G = 6.673 \cdot 10^{-11} \frac{m^3}{\mathrm{kg} s^2}$ is the gravitational constant, m_1, m_2 are the masses of the bodies and $\mathbf{x}_1, \mathbf{x}_2$ are their positions in space.
The transformation to the variables

$$\mathbf{r} := \mathbf{x}_1 - \mathbf{x}_2, \tag{6}$$

$$\mathbf{R} := \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}, \tag{7}$$

leads to the formulation

$$\frac{\partial^2 \mathbf{R}}{\partial t^2} = 0, \tag{8}$$

$$\frac{\partial^2 \mathbf{r}}{\partial t^2} = -GM \frac{\mathbf{r}}{||\mathbf{r}||_2^3}, \tag{9}$$

where R is the center of mass, r the relative distance between the bodies and $M := m_1 + m_2$ is the total mass. If one mass is magnitudes larger than the other, its motion can be neglected and the vector $\mathbf{r}$ describes the motion of the smaller body w.r.t to the larger (resting) body. The mass of the earth is $m_E = 5.97 \cdot 10^{24}$kg.
Tasks/Questions:

(a) Compute the product $m_E \cdot G$ in units $\frac{\text{km}^3}{h^2}$.

(b) Consider a two body system consisting of the earth and a space craft with neglectable mass. Reformulate the system of three ODE of second order (i.e. containing second derivatives w.r.t time) into a system of six ODE of first order (i.e. containing only first derivatives w.r.t time). This can be done introducing an auxiliary variable

$$\mathbf{v} := \frac{\partial \mathbf{r}}{\partial t}, \tag{10}$$

where the physical interpretation of $\mathbf{v}$ is the velocity of the space craft. Use the VectorTrajectoryPlotter and the SimplePlotter in order to provide visualizations of the spatial motion using the initial position $\mathbf{r}(0)^T = (42164\text{km}, 0, 0)$, $t \in [0, 48h]$, an initial velocity $\mathbf{v}(0)^T = s \cdot (0, 1, 0)$ and $s = 4000\text{km/h}, 11038\text{km/h}, 14000\text{km/h}$

(c) What is special qualitative behaviour of the case 11038km/h?

Exercise 4 (Karma Points, just for fun)
Optimize the `SimplePlotter`: introduce coordinate axes, correct plotting orientation to mathematical coordinates (i.e. positive y-direction pointing upwards) and markers for start- and endpoint of the trajectory. Add any feature you miss in the plotter.

Remark: Send your implemented source code (.vrlp file) and the answers to the questions as plain text in an email. Append the pdfs produced with the VectorTrajectoryPlotter and pngs produced witht the SimplePlotter to the email.
Send your solution to `practical.sim1@gcsc.uni-frankfurt.de` until Monday, 21.11.2016, 10h.