Goethe-Center for Scientific Computing (G-CSC)
Goethe-Universität Frankfurt am Main

Modelling and Simulation I

(Practical SIM1, WS 2016/17)

M. Hoffer, Dr. S. Reiter, Dr. A. Vogel

Exercise sheet 2 (Due: Mo., 14.11.2016, 10h)

Exercise 1 (6P + 6P)
Implement the following two explicit one-step methods in order to solve the ordinary differential equation (ODE)

$$\begin{cases} \text{Find } u : [t_0, t_n] \mapsto \mathbb{R}, \text{ such that} \\ \frac{\partial}{\partial t} u(t) = f(t,u) \qquad \text{on } [t_0, t_n], \\ u(t_0) = u_0, \end{cases} \tag{1}$$

where $u_0 \in \mathbb{R}$ is the start value and $t_0, t_n \in \mathbb{R}$ are the start- and endpoints of the interesting time interval. Both methods use the iterative procedure

$$t^{\text{new}} := t^{\text{old}} + h, \tag{2}$$
$$u^{\text{new}} := u^{\text{old}} + h \cdot \phi(t^{\text{old}}, u^{\text{old}}), \tag{3}$$

to compute the next time point and the corresponding solution value, where $\phi(t,u)$ is the method specific step function.

(a) Implement the explicit Heun method: For a given step size h it is based on the step function

$$\phi(t,u) := \frac{1}{2}\left\{k_1 + k_2\right\}, \tag{4}$$
$$k_1 := f(t,u), \tag{5}$$
$$k_2 := f(t+h, u + hk_1), \tag{6}$$
$$\tag{7}$$

(b) Implement the Runge-Kutta-Scheme of 4th order: For a given step size

h it is based on the step function

$$\begin{aligned}
\phi(t,u) &:= \frac{1}{6}\left\{k_1 + 2\,k_2 + 2\,k_3 + k_4\right\}, && (8)\\
k_1 &:= f(t,u), && (9)\\
k_2 &:= f(t+\frac{h}{2}, u+\frac{h}{2}k_1), && (10)\\
k_3 &:= f(t+\frac{h}{2}, u+\frac{h}{2}k_2), && (11)\\
k_4 &:= f(t+h, u+hk_3). && (12)
\end{aligned}$$

Hint: On the website of the practical you will find an exemplary template that is based on the “FunctionPlotter” example that has been discussed on Thursday, 03.11.2016. Open the “ODE-Template” code and rename the class into “ExplicitHeun” (and “ExplicitRungeKutta” for part (b)), and compile the code. By this, a new component is added to the VRL project and you can now modify the relevant code parts, avoiding rewriting of the whole class framework. Note, that instead of “x”, the user function takes “t” and “u” as function parameters.

Exercise 2 (2P + 2P + 4P)
Consider the following ODE:

$$\begin{cases} \text{Find } u : [0,10] \mapsto \mathbb{R}, \text{ such that} \\ u'(t) = -u \qquad \text{on } [0,10], \\ u(0) = 1. \end{cases} \tag{13}$$

Use the TrajectoryPlotter to visualize simultaneously four trajectories:

(i) Explicit Euler using fixed $h = 0.001$ (an approximately exact Solution),

(ii) Explicit Euler using $h = h^*$,

(iii) Heun-Scheme using $h = h^*$,

(iv) Runge-Kutta-Scheme of 4th order using $h = h^*$.

Tasks/Questions:

(a) Produce a picture as a pdf for $h^* = 0.5, 1, 2$ and hand in the 3 pdfs. (Use right-click on the plotted picture and use “Export”)

(b) For $h^* = 0.5$: Which scheme is more accurate w.r.t to the exact solution? Give a list from most accurate to least accurate.

(c) For $h^* = 2$: Which schemes are still able to represent the exact solution qualitatively? Which schemes are instable (i.e. do not represent the exact solution qualitatively)? Can you explain this?

Hint: In order to plot several trajectories into on picture you can use the +/- button of the TrajectoryPlotter. Make sure that all schemes are executed in the control flow before the plotter is invoked. In order to distinguish the different curves assign appropriate names (labels) to the trajectries.

Remark: Send your implemented source code (.vrlp file) and the answers to the questions as plain text in an email. Append the pdfs produced with the TrajectoryPlotter to the email.
Send your solution to `practical.sim1@gcsc.uni-frankfurt.de` until Monday, 14.11.2016, 10h.