Goethe-Center for Scientific Computing (G-CSC)
Goethe-Universität Frankfurt am Main

Modelling and Simulation I

(Practical SIM1, WS 2016/17)

M. Hoffer, Dr. S. Reiter, Dr. A. Vogel

Exercise sheet 1 (Due: Mo., 7.11.2016, 10h)

Exercise 1 (Modelling using ODEs, 8 P)
Formulate an ODE problem for each of the following settings:

(i) Given are 3 species A, B, C of animals that form a food chain: Animals of type A are herbivore and supposed to be supplied with an infinite resource of food, e.g., grass land. They therefore reproduce themselves with a reproduction rate α. The animals of type B are carnivore and hunt the animals of type A. Whenever they encounter animals of type A they find food, can reproduce themselves with a reproduction rate per animal A given by β and the population A therefore decreases with a rate ρ_A. However, in absence of animals of type A the animals B die with a rate δ_B. The species C exclusively feeds upon species B with a reproduction rate per animal B given by γ, resulting in a reduction of the population B by a rate ρ_B. In absence of animals of type B the animals C die with a rate δ_C. At time $t = 0$ a population size $A = 30, B = 10, C = 2$ is observed.

(ii) Given is a population of bacteria D in a petri dish. In an infinite dish the bacteria could reproduce themselves proportional to the current population size. However, in the finite petri dish, the population cannot exceed the size of 100 and the population change is therefore not only proportional to the current population size but also to the remaining space (i.e., the difference to the maximum). Assume a proportionality constant α and an initial population size of 2.

Exercise 2 (Computing solutions, implicit Euler, 7 P)
Consider the following ODE problem: Find a function $u : [0, \infty] \to \mathbb{R}$, such that

$$\begin{aligned} u(0) &= 1, \\ \frac{\partial u}{\partial t} &= 2u, \qquad \text{for } t \in (0, \infty). \end{aligned}$$

Consider to compute the discrete solution $u_{\Delta t}(t)$ at time $t = 1$ using the discrete points $t_k := k\Delta t$ $(k = 0, 1, 2, ...)$ with $0 \leq t_k \leq 1$ and the implicit Euler method.

(i) What is the exact solution $u(t)$ for this problem? What is the exact value $u(1)$?

(ii) Provide a recursive formula computing $u_{\Delta t}(t + \Delta t)$ from $u_{\Delta t}(t)$ using the implicit Euler scheme for this ODE.

(iii) Provide an explicit formula depending on k and Δt to compute $u(k\Delta t)$ when using the implicit Euler scheme for this ODE.

(iv) Compute the value $u_{\Delta t}(1)$, i.e., the approximate value at $t = 1$, for the time step sizes $\Delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.

Exercise 3 (Consistency, implicit Euler, 5 P)
Remember that a sufficiently smooth function $u : \mathbb{R} \to \mathbb{R}$ allows for a Taylor series expansion given by

$$u(x) = \sum_{k=0}^{n} \frac{u^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{u^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

around point x_0 using a certain ξ between the points x and x_0.

(i) Provide the Taylor series expansion for $x := t, x_0 := t + \Delta t$, and $n := 1$.

(ii) Compute the consistency error $\tau_{\Delta t}(t, u)$ for the implicit Euler method and state your result as an explicit formula using $\Delta t, \xi$ and derivatives of u.

(iii) Determine the consistency order for the implicit Euler method.

Remark: Send your answers to the questions as plain text / pdf / scanned pages via email to `practical.sim1@gcsc.uni-frankfurt.de` until Monday, 7.11.2016, 10h.